ON THE STABILIZATION OF CONTROL SYSTEMS

(O STABILIZATSII SISTEM REGULIROVANIIA)

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This paper deals with a general approach to the description of the possible methods of stabilization of automatic control systems, on the basis of the method of Liapunov functions. It is assumed that the transfer function of the object has $n-1$ poles in the left half plane and one single pole at the origin. A method is given for choosing the parameters which will yield asymptotic stability for systems with variable structure, and having a limiter.

1. Let the control system be described by the system of differential equations with constant coefficients

$$x_i^{\cdot} = a_{i1}x_1 + \ldots + a_{in}x_n \qquad (i = 1, \ldots, n) \qquad \left(( )^{\cdot} = \frac{d}{dt}\right) \tag{1.1}$$

We shall assume that the characteristic equation of the system (1.1) has one root equal to zero and $n-1$ roots with negative real parts.

We shall set the problem of the description of the possible methods of correction of a control system which yield asymptotic stability for any initial perturbation conditions. This problem is interesting since in practice there are often systems for which the transfer function of the controlled object has a simple pole at the origin. From a mathematical point of view, the method of study used here, is analogous to those made when the questions of stability in the initial case of one root equal to zero were studied ([1] p.140).

Furthermore, we shall assume below, that among the minors of order $n-1$, different from 0, of the matrix A of the coefficients of the system, there is at least one composed of coefficients of the $n-1$ last equations of the system (1.1). We shall consider together with the system (1.1) the system

$$x_i^{\cdot} = a_{i1}x_1 + \ldots + a_{in}x_n - u(x_1, \ldots, x_n)\,\varphi_i(x_1, \ldots, x_n) \tag{1.2}$$

where the functions $\varphi_i(x_1, \ldots, x_n)$ are given and the function $u(x_1, \ldots, x_n)$ is subject to determination.

As known, there exists a general linear transformation

$$y = b_1x_1 + \ldots + b_nx_n, \quad y_i = b_{i1}x_1 + \ldots + b_{in}x_n \qquad (i = 1, \ldots, n-1, \ b_n \neq 0)$$

which transforms the system (1.1) into the system of differential equations

$$y^{\cdot} = 0, \qquad y_i^{\cdot} = p_{i1}y_1 + \ldots + p_{i,n-1}y_{n-1} \qquad (i = 1, 2, \ldots, n-1) \tag{1.3}$$

This transformation changes the system (1.2) into the system

$$y^{\cdot} = -u\,(b_1\varphi_1 + \ldots + b_n\varphi_n)$$
$$y_i^{\cdot} = p_{i1}y_1 + \ldots + p_{i,n-1}y_{n-1} - u\,(b_{i1}\varphi_1 + \ldots + b_{in}\varphi_n) \tag{1.4}$$

Since the roots of the characteristic equation of the matrix P composed of the coefficients p_{ik} $(i = 1, \ldots, n-1,\ \ k = 1, \ldots, n-1)$, have negative real parts, then for any negative definite form $w(y_1, \ldots, y_{n-1})$ of the variables $y_1, \ldots, y_{n-1}$ there is a positive definite form $v_1(y_1, \ldots, y_{n-1})$ of the same variables such that on the basis of the system (1.3) we shall get

$$\frac{dv_1}{dt} = \sum_{i=1}^{n-1} \frac{\partial v_1}{\partial y_i}\,(p_{i1}y_1 + \ldots + p_{i,n-1}\,y_{n-1}) = w\,(y_1, \ldots, y_{n-1}) \tag{1.5}$$

We shall consider the quadratic form

$$v\,(y_1, \ldots, y_{n-1}, y) = v_1\,(y_1, \ldots, y_{n-1}) + {}^1/_2\, y^2 \tag{1.6}$$

and shall take its derivative on the basis of the system (1.4).

$$\frac{dv}{dt} = w\,(y_1, \ldots, y_{n-1}) - u \sum_{k=1}^{n} \Big(\sum_{i=1}^{n-1} \frac{\partial v_1}{\partial y_i} b_{ik} + b_k y\Big)\varphi_k \tag{1.7}$$

We shall now impose a complementary limitation on the functions φ_k, by requiring that for $y_1 = y_2 = \ldots = y_{n-1} = 0,\ y \neq 0$ the following inequality be satisfied:

$$b_1\varphi_1 + \ldots\ldots + b_n\varphi_n \neq 0.$$

In this case, condition

$$\operatorname{sign} u = \operatorname{sign} \sum_{k=1}^{n} \Big(\sum_{i=1}^{n-1} \frac{\partial v_1}{\partial y_i}\, b_{ik} + b_k y\Big)\varphi_k \tag{1.8}$$

is a sufficient condition of asymptotic stability of the zero solution of the system (1.2). Since on the right-hand side, in the parentheses, there is a linear form of the variables $y_1, \ldots, y_{n-1}, y$, then this condition, going back to the previous variables, can be rewritten

$$\operatorname{sign} u = \operatorname{sign} \sum_{k=1}^{n} (c_{k1}x_1 + \ldots + c_{kn}x_n)\varphi_k \tag{1.9}$$

It is interesting to consider the case, when the function $u\,(x_1, \ldots, x_n)$, representing the command of the control system, belongs beforehand to a given class of functions ([3] p. 608).

For instance, let the function u satisfy the condition $|u| \leqslant 1$, and furthermore let $\varphi_1 = \varphi_2 = \ldots = \varphi_{n-1} \equiv 0,\ \varphi_n = \varphi$. If, from condition (1.9) we shall search for the function u, which yields the most rapid decrease of the Liapunov function v [2 and 3] along the system trajectories (1.2), we shall get

$$u = \operatorname{sign}\,(c_1x_1 + \ldots + c_nx_n)\,\varphi\,(x_1, \ldots, x_n) \tag{1.10}$$

thus the command u will be a piecewise constant function and for $\varphi \equiv 1$ the system will be a pure relay system.

We shall consider the case in which the function φ is given by the relations

$$\varphi = Kx_n \quad \text{for } |Kx_n| \leqslant H, \qquad \varphi = H \operatorname{sign} x_n \quad \text{for } |Kx_n| > H$$

where K and H are positive constants. From (1.10) there follows that

$$u = \operatorname{sign}\,(c_1x_1 + \ldots + c_nx_n)\, x_n$$

Thus, in the present case we get a correction of the control system with a limiter on the input. If $K = \infty$, we get a pure relay system. For $H = \infty$ and K finite we get a class of control systems (systems with variable structure), studied lately in many works [4 and 5].

It is important to notice that with the method described above for choosing the quantities c_i the systems considered will be asymptotically stable for any positive values of a gain coefficient K and the quantity H, whereas in many other cases, when the transfer function of the system has poles in the right half-plane, the stabilization of the system is achieved by the choice of a sufficiently large value of K [5]. On the other hand, from (1.7) there follows that an increase of K and H yields a better quality of the transient response.

2. Following Liapunov ([1] p. 140) we shall now give a method of construction of a linear transformation which changes the system (1.1) into the system (1.3). We shall determine first the quantity

$$y = b_1 x_1 + \ldots + b_n x_n \ (b_n = 1)$$

in such a way that on the basis of the system (1.1) the equality

$$y^{\cdot} = \sum_{i=1}^{n} \sum_{k=1}^{n} b_i a_{ik} x_k = 0 \qquad (2.1)$$

is satisfied.

Condition (2.1) leads to the system of equations

$$a_{1k} b_1 + \ldots + a_{nk} b_n = 0 \qquad (k = 1, \ldots, n) \qquad (2.2)$$

which, obviously, has the required solution, since the rank of the matrix A is equal to $n - 1$. Furthermore, we assume

$$y_i = x_i - d_i y \qquad (i = 1, \ldots, n-1)$$

and get the coefficients d_i from the conditions by which $y_i^{\cdot}$ does not depend upon the variable y. Easy computations yield the system

$$y_i^{\cdot} = \sum_{k=1}^{n-1} (a_{ik} - a_{in} b_k) y_k + \left[\sum_{k=1}^{n-1} (a_{ik} - a_{in} b_k) d_k + a_{in} \right] y$$
$$(i = 1, \ldots, n-1) \qquad (2.3)$$

Equating to zero the coefficients of y, we get for the determination of d_k the system

$$\sum_{k=1}^{n-1} (a_{ik} - a_{in} b_k) d_k + a_{in} = 0 \quad (i = 1, \ldots, n-1) \qquad (2.4)$$

which has a solution since its determinant is not equal to zero. In fact, the transformation from coordinates $x_1, \ldots, x_n$ to coordinates $y, y_1, \ldots, y_{n-1}$ does not change the rank of the matrix composed of the coefficients of the system. The rank of the matrix composed of the coefficents of the system (2.3) (together with the equation $y^{\cdot} = 0$), is equal to $n - 1$, whereupon $(a_{ik} - a_{in} b_k) \neq 0$, since if it were not true, the characteristic equation of the system would have, at least, one more root equal to zero.

Thus, the transformation searched for has the form

$$y = b_1 x_1 + \ldots + b_n x_n, \qquad y_i = x_i - d_i (b_1 x_1 + \ldots + b_n x_n)$$
$$(i = 1, 2, \ldots) \qquad (2.5)$$

where d_i and b_k are determined from the systems (2.2) and (2.4).

We shall consider furthermore the case in which the control system is described by Equation

$$L(p) x = -u(x, x^{\cdot}, x^{\cdot\cdot}, \ldots, x^{(n-1)}) \varphi(x, x^{\cdot}, x^{\cdot\cdot}, \ldots, x^{(n-1)}) \qquad (2.6)$$
$$\left(L(p) = p^n + a_1 p^{n-1} + \ldots + a_{n-1} p, \quad p = \frac{d}{dt} \right)$$

whereupon Equation

$$\lambda^{n-1} + a_1 \lambda^{n-2} + \ldots + a_{n-1} = 0$$

has only roots with negative real parts.

Equation (2.6) can be represented by the system

$$x_i^{\cdot} = x_{i+1}(i = 1, \ldots, n-1), \qquad x_1 = x$$

$$x_n^{\cdot} = -a_{n-1}x_2 - \ldots - a_1 x_n - \qquad u(x_1, \ldots, x_n)\,\varphi(x_1, \ldots, x_n) \tag{2.7}$$

From the system of equations (2.2) we have $b_i = a_{n-1}$ for $i = 1, \ldots, n-1$, $b_n = 1$.

Let now $w(x_2, \ldots, x_n)$ be any negative definite function of the variables $x_2, \ldots, x_n$. It is possible to find such a negative definite quadratic form $v_1(x_2, \ldots, x_n)$ of the same variables, the full derivative of which will be equal to w on the basis of the system

$$x_i^{\cdot} = x_{i+1}\ (i = 2, \ldots, n-1), \qquad x_n^{\cdot} = -a_{n-1}x_2 - \ldots - a_1 x_n \tag{2.8}$$

Obviously, the function v_1 exists since the roots of the characteristic equation of the system (2.8) have negative real parts.

We shall consider the function

$$v(y, x_2, \ldots, x_n) = v_1(x_2, \ldots, x_n) + {}^1/_2\, y^2 \qquad \left(y = \sum_{i=1}^{n-1} a_{n-i}x_i + x_n\right)$$

Taking the derivative of the function v, on the basis of the system (2.7) we get

$$\frac{dv}{dt} = w - \left(\frac{\partial v_1}{\partial x_n} + \sum_{i=1}^{n-1} a_{n-i}x_i + x_n\right) u(x_1, \ldots, x_n)\,\varphi(x_1, \ldots, x_n)$$

Thus, in the present case, we shall get as sufficient conditions of asymptotic stability the relations

$$\varphi(x_1, 0, \ldots, 0) \neq 0 \qquad \text{for } x_1 \neq 0$$

$$\operatorname{sign} u = \operatorname{sign} \varphi(x_1, \ldots, x_n) \left(\frac{\partial v_1}{\partial x_n} + \sum_{k=1}^{n-1} a_{n-k}x_k + x_n\right)$$

Now, if we assume $\varphi(x_1, \ldots, x_n) = Kx_1$ and we search the control function u among the class of functions which satisfy the condition $|u| \leqslant 1$, proceeding from the requirement of the minimum of dv/dt, we get

$$u = \operatorname{sign} x_1 \left(\frac{\partial v_1}{\partial x_n} + \sum_{k=1}^{n-1} a_{n-k}\, x_k + x_n\right) \tag{2.9}$$

3. Now, we shall consider the problems of the stabilization of a control system with a slightly different point of view. Let us assume in the system (2.7)

$$\varphi(x_1, \ldots, x_n) = Kx_1 \qquad (K > 0), \qquad u = \operatorname{sign} x_1 \sigma(r, x)$$

$$\sigma(r, x) = c_{n-1}(r)\, x_1 + c_{n-2}(r)\, x_2 + \ldots + c_0(r)\, x_n \tag{3.1}$$

where r is a real parameter, and the coefficients $c_i(r)$ satisfy the conditions

$$c_0(r) = 1, \quad c_{i+1}(r) = rc_i(r) + a_{i+1} \qquad (i = 0, 1, \ldots, n-2), \quad |r\, c_{n-1}(r)| \leqslant K \tag{3.1}$$

It is known [6] that conditions (3.1) guarantee the presence of a slipping regime in any point of the hyperplane $\sigma(r, x) = 0$.

Introducing the new coordinates $y_i = x_i\ (i = 1, 2, \ldots, n-1)$, $y_n = \sigma(r, x)$, the system (2.7) yields the system

$$\frac{dy_i}{dt} = y_{i+1} \qquad (i = 1, \ldots, n-2), \quad \frac{dy_{n-1}}{dt} = -\sum_{i=1}^{n-1} c_{n-i}(r)\, y_i + y_n$$

$$\frac{dy_n}{dt} = ry_n - \alpha K|y_1| \operatorname{sign} y_n \qquad \left(\alpha = 1 + \frac{r}{K} c_{n-1}(r) \operatorname{sign} y_1 y_n\right) \tag{3.2}$$

From condition (3.1) there follows $0 < \alpha < 2$.

It is known [4], that the motion in the hyperplane of slipping is described by the system

$$\frac{dy_i}{dt} = y_{i+1} \qquad (i = 1, \ldots, n-2), \qquad \frac{dy_{n-1}}{dt} = -\sum_{i=1}^{n-1} c_{n-i}(r)\, y_i \qquad (3.3)$$

Let $r < 0$ and the zero solution of the system (3.3) be asymptotically stable. It can be shown that in such a case the zero solution of the system (3.2) and consequently that of the system (2.7) (for $\varphi = Kx_1$, $u = \operatorname{sign} x_1\sigma(r, x)$) will be asymptotically stable for any initial perturbation conditions.

In fact, on the basis of the well known theorem by Liapunov ([1] p. 107), there exists a positive definite quadratic form $v_1(y_1, \ldots, y_{n-1})$, the total derivative of which is, on the basis of system (3.3), equal to the quadratic form $w = -y_1^2 - \ldots - y_{n-1}^2$.

We shall consider the function

$$v(y_1, \ldots, y_n) = v_1(y_1, \ldots, y_{n-1}) + \tfrac{1}{2} A y_n^2 \qquad (A > 0)$$

The time derivative of the function v, taken on the basis of the system (3.2) has the form

$$\frac{dv}{dt} = -\sum_{i=1}^{n-1} y_i^2 + \frac{\partial v_1}{\partial y_{n-1}} y_n + Ary_n^2 - \alpha AK\,|y_1 y_n| \qquad (3.4)$$

Since $r < 0$, there follows from Sylvester's criterion, that for a sufficiently large positive value of A, the quadratic form on the right-hand side of Equation (3.4) will become negative definite. Since $\alpha > 0$, then the Liapunov's theorem on asymptotic stability can be now applied.

We shall notice that in formulation of the last result, unlike in the previous one, the real parts of the roots of Equation $\lambda^{n-1} + a_1\lambda^{n-2} + \ldots + a_{n-1} = 0$ were not required to be negative. However, if this requirement is considered fulfilled, then, on the basis of the second consequence of Theorem 2 of [7], there exists a negative value of r which yields asymptotic stability in the hyperplane of slipping. Therefore, in the same case, the property of the global asymptotic stability of the zero solution of the system (2.7) takes place for the chosen method of control.

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